# Inner Product Space

Department of Computer Engineering

Sharif University of Technology

Hamid R. Rabiee    rabiee@sharif.edu

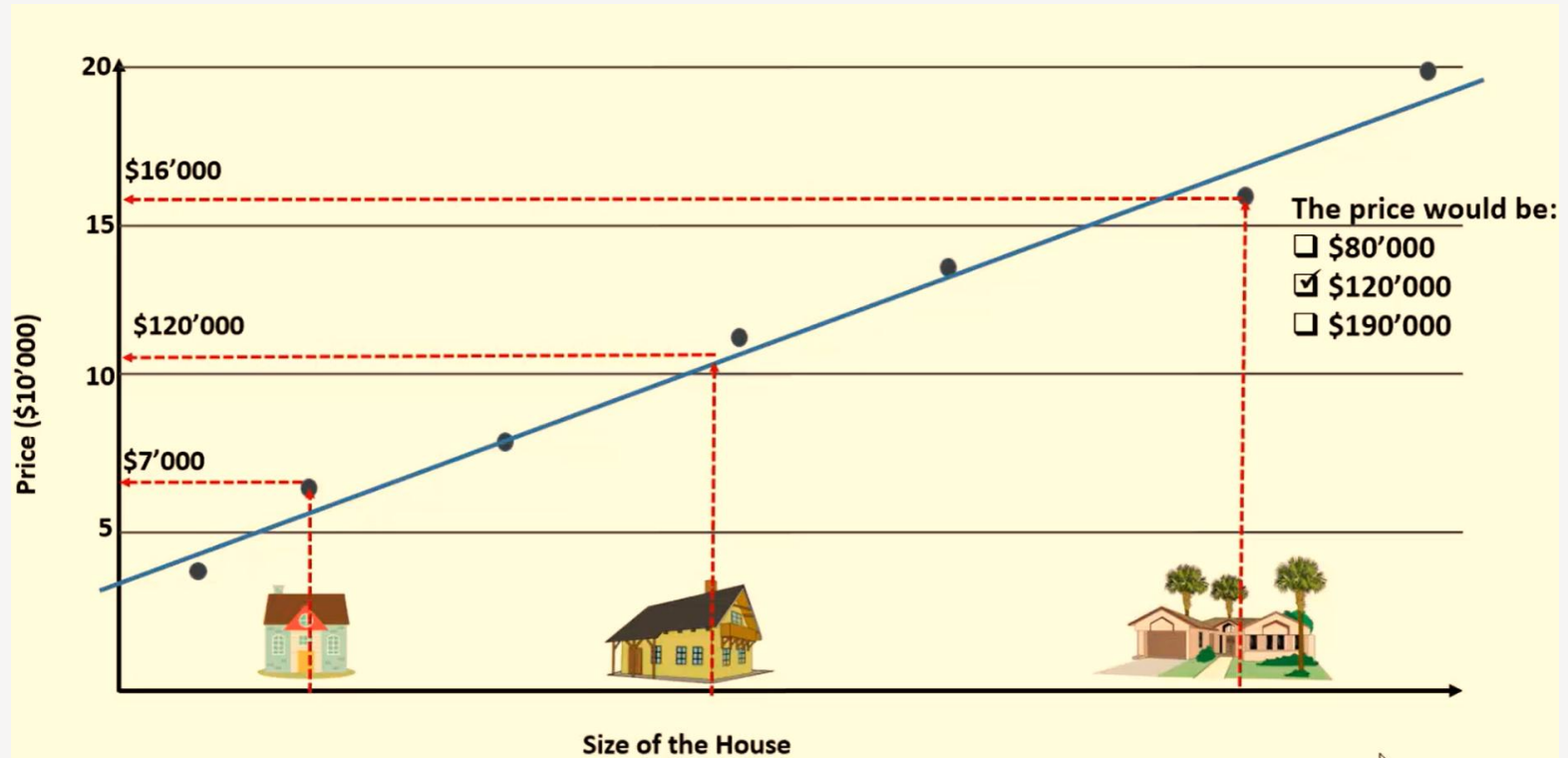Maryam Ramezani    maryam.ramezani@sharif.edu

# Table of contents

# 01

# Introduction

# Least Squares Error Correction

# Least Squares Error Correction
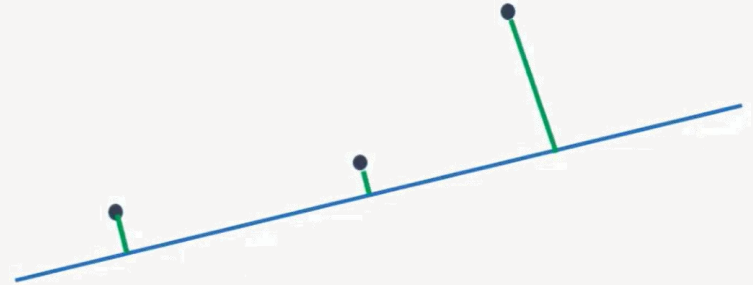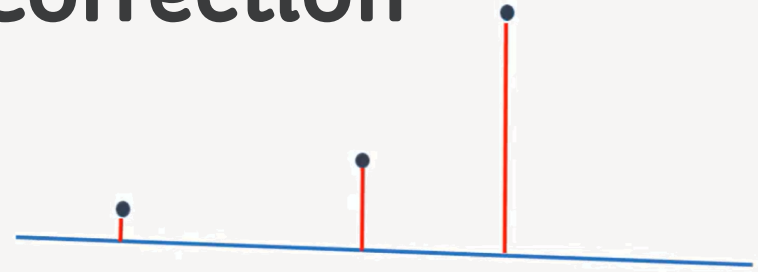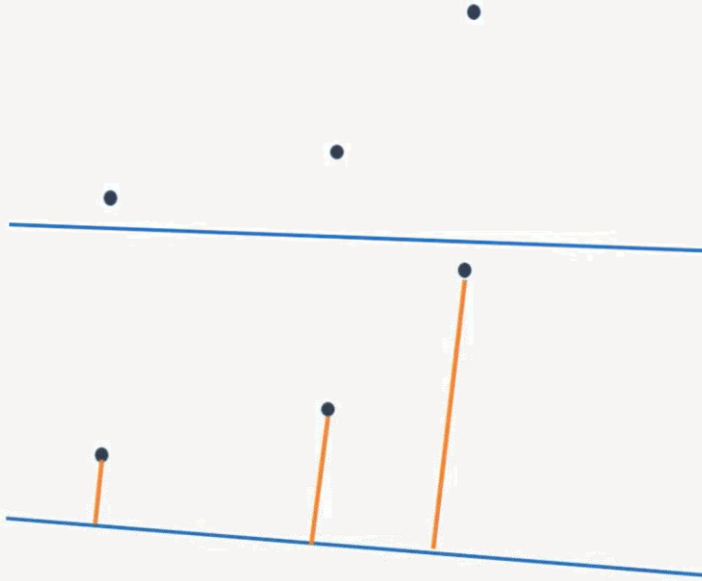


Error 1: ▬▬▬▬▬▬▬▬▬▬▬▬

Error 2: ▬▬▬▬▬▬▬▬▬▬▬▬▬▬▬▬

Error 3: ▬▬▬▬▬▬▬▬

# 02

# Linear Form

# What are Linear Functions?

❑ $f: R^n \rightarrow R$ means that $f$ is a function that maps real $n$-vectors to real numbers

❑ $f(x)$ is the value of function $f$ at $x$ ($x$ is referred to as the argument of the function).

❑ $f(x) = f(x_1, x_2, \dots, x_n)$: where $x_1, x_2, \dots, x_n$ are arguments

| Definition |
|---|
| A function $f: R^n \rightarrow R$ is linear if it satisfies the following two properties: <br><br> ❑ **Additivity**: For any $n$-vector $x$ and $y$, $f(x + y) = f(x) + f(y)$ <br><br> ❑ **Homogeneity**: For any $n$-vector $x$ and any scalar $\alpha \in R$: $f(\alpha x) = \alpha f(x)$ |

# Superposition property:

**Definition**

Superposition property:

$$f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$$

**Note**

❑ A function that satisfies the superposition property is called linear

# What are Linear Functions?

❑  If a function f is linear, superposition extends to linear combinations of any number of vectors:

$$f(\alpha_1 x_1 + \cdots + \alpha_k x_k) = \alpha_1 f(x_1) + \cdots + \alpha_k f(x_k)$$

# Inner product is Linear Function?

## Theorem 1

A function <span style="color:red">defined as the inner product</span> of its argument with some fixed vector <span style="color:red">is linear</span>.

<span style="color:magenta">Proof?</span>

$$f(x) = a^T x = a_1 x_1 + a_2 x_2 + \cdots + a_n x_n$$

# What are Linear Functions?

## Theorem 2

If a function is linear, then it can be expressed as the inner product of its argument with some fixed vector.

Proof?

# What are Linear Functions?

**Theorem 3**

The representation of a linear function $f$ in a specific basic as:
$f(x) = a^T x$ is <span style="color:red">unique</span>, which means that there is only one vector $a$ for which $f(x) = a^T x$ holds for all $x$.

Proof?

# Linear Form Examples

**Example**

- Is average a linear function?
- Is maximum a linear function?

# 03

# Bilinear Form

# Bilinear Form over a real vector space

## Definition

Suppose $V$ and $W$ are vector spaces over the same field $\mathbb{F}$. Then a function $f: V \times W \rightarrow \mathbb{F}$ is called a **bilinear form** if it satisfies the following properties:

    a) It is linear in its first argument:
       i.  $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and
      ii.  $f(c\mathbf{v_1}, \mathbf{w}) = cf(\mathbf{v_1}, \mathbf{w})$ for all $c \in \mathbb{F}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.
    b) It is linear in its second argument:
       i.  $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and
      ii.  $f(\mathbf{v}, c\mathbf{w_1}) = cf(\mathbf{v}, \mathbf{w_1})$ for all $c \in \mathbb{F}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

# Bilinear Form

**Note**

Let $V$ be a vector space over a field $\mathbb{F}$. Then the **dual** of $V$, denoted by $V^*$, is the vector space consisting of all linear forms on $V$.

**Example**

Let $V$ be a vector space over a field $\mathbb{F}$. Show that the function $g: V^* \times V \to \mathbb{F}$ defined by
$$g(f, \mathrm{v}) = f(\mathrm{v}) \text{ for all } f \in V^*, \mathrm{v} \in V$$
is a bilinear form.

# Symmetric Bilinear Form

**Definition**

A **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space $V$ is called symmetric if for all $v, w \in V$:
$$f(v, w) = f(w, v)$$

# Bilinear Form arises from a matrix

## Theorem 4

Every **bilinear form** function $f: V \times V \rightarrow \mathbb{F}$ over a real vector space $V$ arises from a matrix for all $v, w \in V$:

$$f(v, w) = v^T A w$$

Proof?

# Associated Matrices

If $V$ is a finite-dimensional vector space, $B = \{b_1, \ldots, b_n\}$ is a basis of $V$, and $f: V \times V \to \mathbb{F}$ be a **bilinear form** function the associated matrix A of $f$ with respect to $B$ is the matrix $[f]_B \in \mathbb{F}^{n \times n}$ whose $(i, j)$-entry is the value $f(b_i, b_j)$.

$$f(v, w) = v^T A w = v^T [f]_B w$$

$$[f]_{\mathcal{B}} = \begin{pmatrix} f(b_1, b_1) & \cdots & f(b_1, b_n) \\ \vdots & & \vdots \\ f(b_n, b_1) & \cdots & f(b_n, b_n) \end{pmatrix}$$

# Associated Matrices

## Example

For the bilinear form $f\left(\begin{bmatrix} a \\ b \end{bmatrix}, \begin{bmatrix} c \\ d \end{bmatrix}\right) = 2ac + 4ad - bc$ on $\mathbb{F}^2$, find $[f]_B$ for basis $B = \{\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 4 \end{bmatrix}\}$ and $[f]_P$ for basis P= $\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\}$

# 04

# Bilinear Form Over Complex Vector Space

# Bilinear Form over a complex vector space

## Definition

Suppose $V$ and $W$ are vector spaces over the same field $\mathbb{C}$. Then a function $f: V \times W \rightarrow \mathbb{C}$ is called a **bilinear form** if it satisfies the following properties:

  a) It is linear in its first argument:
   i.  $f(\mathbf{v_1} + \mathbf{v_2}, \mathbf{w}) = f(\mathbf{v_1}, \mathbf{w}) + f(\mathbf{v_2}, \mathbf{w})$ and
   ii. $f(\lambda \mathbf{v_1}, \mathbf{w}) = \lambda f(\mathbf{v_1}, \mathbf{w})$ for all $\lambda \in \mathbb{C}, \mathbf{v_1}, \mathbf{v_2} \in V$, and $\mathbf{w} \in W$.
  b) It is conjugate linear in its second argument:
   i.  $f(\mathbf{v}, \mathbf{w_1} + \mathbf{w_2}) = f(\mathbf{v}, \mathbf{w_1}) + f(\mathbf{v}, \mathbf{w_2})$ and
   ii. $f(\mathbf{v}, \lambda \mathbf{w_1}) = \bar{\lambda} f(\mathbf{v}, \mathbf{w_1})$ for all $\lambda \in \mathbb{C}, \mathbf{v} \in V$, and $\mathbf{w_1}, \mathbf{w_2} \in W$.

# Bilinear Form over a complex vector space

| Bilinear forms on $\mathbb{R}^n$ | Bilinear forms on $\mathbb{C}^n$ |
|:---:|:---:|
| <u>Linear</u> in the first variable | <u>Linear</u> in the first variable |
| <u>Linear</u> in the second variable | <u>Conjugate linear</u> in the second variable |

# 05

# Inner Product

# Inner product over real vector space

**Definition**

An inner product is a positive-definite symmetric bilinear form.

An inner product on $V$ is a function $\langle,\rangle: V \times V \rightarrow \mathbb{R}$ such that $v, w \in V, c \in \mathbb{R}$:

1.  $\langle v, v \rangle = 0$ if and only if $v = 0$.

2.  $\langle w, v \rangle = \langle v, w \rangle$.

3.  $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all $u, v, w \in V$.

4.  $\langle cw, u \rangle = c\langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.

5.  $\langle v, v \rangle \geq 0$ for all $v \in V$.

# Inner Product

**Why for bilinear form I wrote just two properties instead of four properties?**

❑ Using properties (2) and (4) and again (2)

$$\langle w, cu \rangle = \langle cu, w \rangle = c\langle u, w \rangle = c\langle w, u \rangle$$

❑ Using properties (2), (3) and again (2)

$$\langle w, u + v \rangle = \langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle = \langle w, u \rangle + \langle w, v \rangle$$

1. $\langle v, v \rangle = 0$ if and only if $v = 0$.

2. $\langle w, v \rangle = \langle v, w \rangle$.

3. $\langle u + v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$ for all u, $v, w \in V$.

4. $\langle cw, u \rangle = c\langle w, u \rangle$ for all $u, w \in V$ and $c \in \mathbb{R}$.

5. $\langle v, v \rangle \geq 0$ for all $v \in V$.

# Inner Products

**Note**

- ❑ For $v \in V, \langle 0, v \rangle = 0, \langle v, 0 \rangle = 0$.

# General Inner product

**Definition**

Suppose that $\mathbb{F} = \mathbb{R}$ or $\mathbb{F} = \mathbb{C}$ and that $V$ is a vector space over $\mathbb{F}$. Then an **inner product** on $V$ is a function $\langle \cdot, \cdot \rangle : V \times V \rightarrow \mathbb{F}$ such that the following three properties hold for all $c \in \mathbb{F}$ and all $\mathbf{v}, \mathbf{w}, \mathbf{x} \in V$:

   a) $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ (conjugate symmetry)

   b) $\langle \mathbf{v} + c\mathbf{x}, \mathbf{w} \rangle = \langle \mathbf{v}, \mathbf{w} \rangle + c \langle \mathbf{x}, \mathbf{w} \rangle$ (linearity)

   c) $\langle \mathbf{v}, \mathbf{v} \rangle \geq 0$, with equality if and only if $\mathbf{v} = \mathbf{0}$. (pos. definiteness)

# Inner Products for vectors

**Note**

❑ The standard inner product between vectors is: ($x, y \in \mathbb{R}^n$)

$$\langle x, y \rangle = x^T y = \sum x_i y_i$$

❑ The function $\langle \cdot, \cdot \rangle : \mathbb{C}^n \times \mathbb{C}^n \rightarrow \mathbb{C}$ defined by

$$\langle v, w \rangle = v^* w = \sum_{i=1}^{n} \bar{v}_i \, w_i$$

for all $v, w \in \mathbb{C}^n$ is an inner product on $\mathbb{C}^n$.

# Inner Products for matrices

## Note

The standard inner product between two matrices is: $(X, Y \in \mathbb{R}^{m \times n})$

$$\langle X, Y \rangle = trace(X^T Y) = \sum_i \sum_j X_{ij} Y_{ij}$$

## Example

Find the inner products of following matrices:

$$U = \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}, \qquad V = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

# Inner Product for functions

## Note

Let $a < b$ be real numbers and let $C[a, b]$ be the vector space of continuous functions on the real interval $[a, b]$. The function $\langle \cdot, \cdot \rangle : C[a, b] \times C[a, b] \to \mathbb{R}$ defined by

$$\langle f, g \rangle = \int_a^b f(x)g(x)\, dx \qquad \text{for all} \qquad f, g \in C[a, b]$$

is and inner product on $C[a, b]$.

# Inner Product for polynomials

**Note**

❑ For $p(x)$ and $q(x)$ with at most degree $n$:

$$\langle p(x), q(x)\rangle = p(0)q(0) + p(1)q(1) + \cdots + p(n)q(n)$$

❑ For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = p(0)q(0) + \int_{-1}^{1} p'q'$

❑ For $p(x)$ and $q(x)$: $\langle p(x), q(x)\rangle = \int_{0}^{\infty} p(x)q(x)e^{-x}dx$

# 06

# Inner Product Space

# Inner product space

**Definition**

An inner product space is a finite-dimensional real or complex vector space $V$ along with an inner product on $V$.

Euclidean Space   Unitary Space

# Resources

❑ Chapter 8: Kenneth Hoffman and Ray A. Kunze. Linear Algebra. PHI Learning, 2004.

❑ Chapter 6: Sheldon Axler, Linear Algebra Done Right, 2024.

❑ Chapter 1: David C. Lay, Steven R. Lay, and Judi J. McDonald. Linear Algebra and Its Applications. Pearson, 2016.

❑ Chapter 2: David Poole, Linear Algebra: A Modern Introduction. Cengage Learning, 2014.

❑ Chaper1: Gilbert Strang. Introduction to Linear Algebra. Wellesley-Cambridge Press, 2016.